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Random cellular froths in spaces of any dimension and curvature

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Abstract. Froth is a random partition of a D-dimensional space by cells. This assembly of cells obeys two fundamental laws: Euler's relation and the condition of maximum vertex figure, imposed by geometry and by topological stability, respectively. These two conditions generate a set of relations between the variables that fully characterize the system topologically. The number of degrees of freedom of the system and a set of useful independent variables, the 'even valences', have been found. The influence of the space dimension and curvature on the range of variability of these valences is discussed and, up to $D = 5$, the regions in valence space corresponding to differently curved froths are calculated explicitly.

1. Introduction

A froth is a random division of space by cells. Consider a froth in D-dimensional space: a cell of the D-froth is a bounded volume, a D-dimensional polytope (polyhedra in three dimensions, polygon in two dimensions and segment in one dimension). The boundary of each D-cell is itself a $(D-1)$-froth constituted by $(D-1)$-polytopes. (A three-dimensional froth is made of 3D polyhedra. Their surface is a 2D elliptic froth whose 'cells' are the interfaces between the original cells.) Cells are convex topological D-dimensional polytopes separated by 'interfaces' which are themselves $(D-1)$-dimensional polytopes, etc.

The D-dimensional froth is therefore a graded topological random set: it contains C_D D-polytopes, the cells, which fill space which can be Euclidean, elliptic or hyperbolic. Each cell is itself an elliptic $(D-1)$-froth, whose convex 'cells' are $(D-1)$-polytopes which are the interfaces bounding the original cell and separating it from its topological neighbours. Each 'interface' of the $(D-1)$-froth is an elliptic $(D-2)$-froth of $(D-2)$-polytopes, which are the common edges of the interfaces separating the cells from their neighbours. The graded topological set terminates with the edges, segments or convex 1-polytopes, trivially bounded by two vertices or 0-polytopes. All homology groups are trivial. All cycles are bounding, faces are disconnected by cutting a closed contour of edges, etc.

Thus, the D-froth is constituted of C_D cells, C_{D-1} 'interfaces', …, C_2 facets, C_1 edges and C_0 vertices, and each element is itself an elliptic froth in its own dimension. The elements are related through Euler relations, which are, for elliptic froths [1]:

$$(-1)^D C_D - (-1)^{D-1} + \cdots + C_2 - C_1 + C_0 = \begin{cases} 2 & D \text{ even} \\ 0 & D \text{ odd}\,. \end{cases} \qquad (1)$$

An Euler relation is satisfied for each froth of the graded topological set. The whole froth can fill a Euclidean space, in which case the right-hand side of Euler's relation equals 1 (one D-cell in the elliptic D-froth corresponds to the cell at infinity in the Euclidean D-froth, and is not counted). In even D, if the original D-froth is hyperbolic, the right-hand side of Euler's relation (the Euler–Poincaré characteristic) is even and negative or zero. The numbers C_i of different elements of the froth are connected by valences $n_{i,j}$

$$n_{j,i}C_i = n_{i,j}C_j \qquad i, j = 0, 1, 2, \ldots, D. \tag{2}$$

Let $i < j$: $n_{j,i}$ is an incidence number, counting the numbers of j-simplices incident on each, lower i-simplex; $n_{i,j}$ is a coordination, denoting the number of lower i-simplices bounding each j-simplex. For example, $n_{2,1}$ is the number of facets incident on an edge and $n_{1,2}$ is the number of edges per facet. In a hexagonal lattice, $n_{2,1} = 2$ and $n_{1,2} = 6$ (any edge separates two facets, any face has six edges).

In random froths, all incidence numbers are fixed by randomness. In a random froth, the randomness fixes all the incidence numbers at its lowest possible value. This is a consequence of the topological stability: any configuration with higher incidence number can be split, by infinitesimal transformations, in a certain number of configurations with minimum incidence number (for example, in $D = 2$ a four-corner vertex can be split into two three-corner vertex through an infinitesimal deformation). The incidence numbers in froths are

$$n_{j,i} = \begin{pmatrix} D+1-i \\ j-i \end{pmatrix} \tag{3}$$

there are $(D+1)$ edges and $\binom{D+1}{2}$ faces incident on each vertex, D faces incident on each edge, etc. These identities are due to randomness or topological stability (the four-corner boundary between Utah, Colorado, New Mexico and Arizona is not topologically stable; it can be split into two stable three-corner boundaries by an infinitesimal deformation). Stability is also dynamical, as befits an energy carried by interfaces (surface tension) [2]. Interface tension is also what makes the various polytopes convex and their faces as flat as possible. In this respect a froth consisting of convex cells is more restricted than a scaffolding. A diamond lattice scaffolding has cells which are not convex and interfaces which are not planar. The froth formed by dipping the diamond scaffolding into a soap solution will be very different. Similarly, the scaffolding of Schwarz's gyroid (a 3D scaffolding with incidence $n_{2,1} = 3$ [3]) does not define convex cells and has twisted interfaces. Another way of stating convexity is that two hedgehogs of edges incident on any two neighbouring vertices are as eclipsed as possible. In particular, $n_{D,(D-1)} = 2$ (an interface separate two cells in the froth). By convention, $n_{i,i} = 1$ is also an incidence number.

By contrast, the coordination numbers $n_{i,j}$ for $i < j$ are random variables in the froth, except for $n_{0,1}$ (every edge is bounded by two vertices). Their averages are severely restricted by relationships imposed by space-filling ($\langle n_{1,2}\rangle = \langle n_{0,2}\rangle = 6$ for a two-dimensional Euclidean froth is the consequence of Euler relation. Furthermore the shape of nearest neighbours' cells are correlated in maximally random froths—a consequence of maximum entropy inference—through the Aboav–Weaire [4, 5] and Peshkin [6] laws [2, 7]. Similar correlations caused by maximum randomness should also occur in froths of higher, even dimensions, but they will not be discussed here).

The main problem discussed in this paper is the organization of the D-dimensional froth. How many average coordinations $\langle n_{i,j}\rangle$ are independent random variables and how many are related further by the condition of filling a Euclidean space? In a word, what is the dimensionality of the random homological [8] complex of cells, interfaces, ..., edges and vertices?

D-dimensional froths have a few practical applications for $D \geqslant 4$, and in non-Euclidean spaces for $D \geqslant 3$. For the latter, one should note that the froth is the topological dual of a packing, of atoms for example, and that three-dimensional packings in positively curved 3D space are very realistic models for amorphous metals [9, 10], quasicrystals [11] and the crystalline alloy phases called tetrahedrally-closed-packed or Frank–Kasper phases [12]. The only physical relevance of froths in more than three dimensions is in information theory [13]. Notably, the most economical storage of data in a memory can be made by partitioning space into cells containing binary information. Efficient updating of this memory is made by cell division, and the problem is to find the best dimensionality D of the initial froth so that all cells affected by the updating are neighbours in the D-froth. Such updating is efficient and much more economical and organized than simply adding a dimension. Note that a different organization of the data and their updating has been suggested and studied by Nadal [14], based on the Hopfield spin-glass model of addressable and adjustable memory. There, updating is made by partition through hyperplanes of the whole structure, rather than by division of a single cell.

In this paper, we call D-froth a space-filling random packing of cells which, in a space of dimensionality D, satisfies the Euler law [15] and the condition of structural stability (or maximal randomness). This condition is the topological rule of a vertex figure with minimum coordination number (as discussed in section 2).

In section 3, we show that the state of the D-dimensional froth can be characterized using a set of variables, the 'valences' $X_k = \langle n_{k-1,k} \rangle$ with $(k = 1, \ldots, D)$, which are the average number of the $(k-1)$-dimensional elements surrounding a k-dimensional cell. These variables are not all independent and, in particular, we have found that the valences with k odd are related to the valences with k even. Furthermore, it is shown that valences with k even constitute a complete set of variables.

The curvature of space is related to the statistical properties of the froth. In section 4 this relation is discussed. We calculate explicitly, up to $D = 5$, the restricted range of independent valences for a Euclidean D-froth.

Finally, in section 5, the possibility of constructing froths in any dimension is discussed and several examples are given.

2. Construction rules for the froth in arbitrary dimension

The cellular structure constituting the D-dimensional froth must satisfy the Euler law and rules of structural stability.

The Euler law must be applied to the D-dimensional cellular system constituting the froth, which can be on an Euclidean, elliptic or hyperbolic manifold. The surface of a D-cell is a cellular structure constituting a $(D-1)$-dimensional froth in an elliptic space, which must also satisfy the Euler law. In the same way, scaling down the dimensions, one has lower-dimensional froths in elliptic spaces that satisfy the Euler law.

The structural stability condition for the system states that the froth must have minimum coordination number for the vertex figures (minimum number of edges incident on a vertex, minimum number of faces incident on a edge, etc). Vertex figures with higher coordination numbers can be transformed into these by infinitesimal deformations.

It is natural to characterize a D-dimensional froth in terms of the numbers C_i of the i-dimensional elements constituting the froth (C_0 number of vertices, C_1 number of edges, C_2 number of faces, C_3 number of cells, etc) and by the average numbers $\langle n_{i,j} \rangle$ of i-dimensional simplices surrounding a j-dimensional one ($\langle n_{0,1}\rangle = 2$ number of vertices surrounding an edge, $\langle n_{1,2}\rangle$ number of edges per faces, etc).

The D-dimensional froth is described by the following set of fundamental equations.

- Euler formula:

$$\sum_{i=0}^{D}(-1)^i C_i = \chi_D \tag{4}$$

where χ_D is the Euler–Poincaré characteristic, which takes the value 1 for a connected Euclidean manifold.

- Euler formulae for the surface of the J-dimensional cells:

$$\sum_{i=0}^{J-1}(-1)^i \langle n_{i,J}\rangle = \chi_{J-1}^{(\mathrm{Elliptic})} \tag{5}$$

with $J = 1, 2, \ldots, D$. Here $\chi_{J-1}^{(\mathrm{Elliptic})}$ is the Euler–Poincaré characteristic for a $(J-1)$ sphere and has a value $\chi_{J-1}^{(\mathrm{Elliptic})} = 1 - (-1)^J$.

- Stability condition for the froth

$$\binom{D+1-i}{j-i} C_i = \langle n_{i,j}\rangle Cj \tag{6}$$

with $i \leqslant j \leqslant D$.

Note that (6) is the standard valence relation $n_{j,i}\ C_i = \langle n_{i,j}\rangle Cj$ in which $n_{j,i}$ is the number of j-simplices incident on any i-simplex. The stability condition for the froth implies that this number always takes its minimum value, i.e. $n_{j,i} = \binom{D+1-i}{j-i}$. For example, $n_{1,0} = D+1$ is the minimum number of edges incident on a vertex in a D-dimensional space.

Note also that (6) is a statistical relation and is only valid for $i, j < D$ (on a surface of a closed j-dimensional simplex) or in the limit $C_D \to \infty$ for $j = D$. Note also that (6) implies $\langle n_{i,i}\rangle = 1$.

3. Definition of a complete set of independent variables

The variables of the problem are the numbers of simplices C_i and the valences $\langle n_{i,j}\rangle$ (with $0 \leqslant i < j \leqslant D$). The total number of these variables is $D + 1 + \frac{1}{2}D(D+1)$. They are not all independent but are related by (4)–(6).

The first step is to find an overcomplete set of variables that can generate all the other variables. The second step will be to find the number of degrees of freedom of the problem and a complete set of independent variables.

The variables C_i are simply related to the valences by (6)

$$C_i = \binom{D+1-i}{D-i}^{-1} \langle n_{i,D}\rangle C_D \tag{7}$$

where the total number of cells of the system C_D can be taken as a extensive parameter of the problem, since we are interested in the statistical properties.

The valences are also related between themselves; from (6) it follows that

$$\langle n_{i,j}\rangle = \binom{D+1-i}{j-i}\frac{C_i}{C_j} = \binom{D+1-i}{j-i}\frac{C_i}{C_{i+1}}\frac{C_{i+1}}{C_{i+2}}\cdots\frac{C_{j-1}}{C_j}. \tag{8}$$

In particular, from (6)

$$\frac{C_i}{C_{i+1}} = \frac{\langle n_{i,i+1}\rangle}{D+1-i}$$

thus, substituting in (8)

$$\langle n_{i,j}\rangle = \frac{(D+1-i)!}{(j-i)!(D+1-j)!}\frac{1}{(D+1-i)}\frac{1}{(D+1-i-1)}\cdots$$
$$\cdots\frac{1}{(D+1-j+1)}\langle n_{i,i+1}\rangle\cdots\langle n_{j-1,j}\rangle \tag{9}$$
$$\langle n_{i,j}\rangle = \frac{1}{(j-i)!}\prod_{k=i+1}^{j} X_k$$

where we have defined $X_k \equiv \langle n_{k-1,k}\rangle$.

Thus, the variables ($X_k, k = 1, 2, \ldots, J \leqslant D$) form overcomplete sets for any subspaces of dimensionality $J \leqslant D$.

These variables X_k are the average number of $(k-1)$-dimensional simplices that form the boundaries of a k-dimensional simplex (X_1 is the number of vertices per edge (always 2), X_2 is the number of edges per face, X_3 is the number of facets per cell, etc). Note that these valences X_k must be rational numbers and must satisfy $X_k \geqslant k+1$ since a k-cell must be bounded by at least $k+1$ neighbours (triangles and tetrahedra are the minimally bounded figures in two and three dimension, respectively).

The variables X_k are not all independent. Substitution of (10) into (5) leads to a relation between these variables

$$\sum_{i=0}^{J-1}(-1)^i\frac{1}{(J-i)!}\prod_{k=i+1}^{J} X_k = \chi_{J-1}^{(\mathrm{Elliptic})} \tag{10}$$

that *for J odd* gives a relation between X_J and the previous valences X_k with $k < J$

$$X_J = \frac{2}{1+\sum_{i=0}^{J-2}\frac{(-1)^i}{(J-i)!}\prod_{k=i+1}^{J-1} X_k}. \tag{11}$$

For $J = 1$ to $J = 2l+1 \leqslant D$ equation (11) gives

$$X_1 = 2$$
$$X_3 = \frac{2}{1+\frac{X_1X_2}{3!}-\frac{X_2}{2}} = \frac{2}{1-\frac{X_2}{6}}$$
$$X_5 = \frac{2}{1+\frac{X_1X_2X_3X_4}{5!}-\frac{X_2X_3X_4}{4!}+\frac{X_3X_4}{3!}-\frac{X_4}{2}}$$
$$= \frac{2}{1-\frac{X_4}{6}\frac{1-\frac{X_2}{5}}{1-\frac{X_2}{6}}} \tag{12}$$
$$X_7 = \frac{2}{1-\frac{X_6}{6}\left\{1-\frac{X_4}{5}\frac{1-\frac{17}{84}X_2}{1-\frac{X_2}{6}}\right\}\Big/\left\{1-\frac{X_4}{6}\frac{1-\frac{X_2}{5}}{1-\frac{X_2}{6}}\right\}}$$
$$\vdots$$
$$X_{2l+1} = \frac{2}{1+\frac{X_1X_2\ldots X_{2l}}{(2l+1)!}-\frac{X_2\ldots X_{2l}}{(2l)!}+\cdots-\frac{X_{2l}}{2}}.$$

For J *even*, equation (10) does not give any new relation between the valences X_k (as proved in appendix A).

The set of equations (12) indicates that all the odd variables X_{2l-1} can be expressed in terms of the even ones. In (12) this expression is given up to $J = 7$. In appendix B this expression is obtained by induction for any J, and it is shown that all X_{2l+1} can be written in the following canonical form:

$$X_{2l+1} = \frac{2}{1 - \frac{X_{2l}}{a_0} H^1_{2(l-1)}} \tag{13}$$

with

$$H^s_{2n} = \frac{1 - \frac{X_{2n}}{a_s} H^{s+1}_{2(n-1)}}{1 - \frac{X_{2n}}{a_0} H^1_{2(n-1)}}. \tag{14}$$

Clearly X_1, X_3, X_5 and X_7 in (12) are canonical, and $H^s_0 = 1$, $H^s_{-2} = 0$ for all s. The coefficients a_s in (14) are numbers, decreasing with increasing s, which are all larger than $a_\infty = \frac{1}{2}\pi^2 = 4.934\,8022$, and converge rapidly to a_∞:

$$a_0 = 6 \qquad a_1 = 5 \qquad a_2 = \frac{84}{17} = 4.941\,18$$
$$a_3 = \frac{153}{31} = 4.935\,484 \qquad a_4 = \frac{3410}{691} = 4.934\,877\,.$$

An additional relation between the valences should be considered when $J = D$. In this case, one must take into account the Euler formula for the whole froth. Dividing equation (4) by C_D, we obtain

$$\sum_{i=0}^{D} (-1)^i \frac{C_i}{C_D} = \sum_{i=0}^{D} \frac{(-1)^i}{(D+1-i)} \langle n_{i,D} \rangle = \frac{1}{C_D}\,. \tag{15}$$

This relation is similar (but not, in general, equal) to (10) with $J = D$, and represents an additional condition on the valences.

Note that in the thermodynamic limit ($C_D \to \infty$), the right-hand side of (15) equals zero. In this limit, substituting (10) into (15) yields a relation between the valences X_k

$$\sum_{i=0}^{D-1} \frac{(-1)^i}{(D+1-i)!} \prod_{k=i+1}^{D} X_k + (-1)^D = 0 \tag{16}$$

where we used $\langle n_{D,D} \rangle = 1$.

This equation provides the following expression for X_D

$$X_D = \frac{2}{1 - 2(-1)^D \sum_{i=0}^{D-2} \frac{(-1)^i}{(D+1-i)!} \prod_{k=i+1}^{D-1} X_k}\,. \tag{17}$$

For D even, equation (17) is a new relation between the previously free valences and X_D. For D odd, equation (17) is equivalent to (11) with $J = D$. This point is discussed in appendix A, where we calculate the number of degrees of freedom of the system. It turns out that, for D odd, equation (17) does not constrain the variables of the problem.

In conclusion, the even valences X_{2l} with $2l < D$ constitute a complete set of variables which determine the statistical topological properties of an Euclidean D-dimensional froth. For example, in $D = 2$, all variables are constrained, $X_2 = 6$. In $D = 3$, we have one variable X_2, as in $D = 4$. In $D = 5$ we have two variables X_2 and X_4, etc.

4. Space curvature and D-dimensional froths

In a Euclidean D-dimensional space the froth is made up of D-dimensional simplices: the cells. The surface of such simplices is a $(D-1)$-dimensional elliptic manifold covered by a froth. Scaling down the dimensions one obtains that the froth is constituted of elliptic lower-dimensional froths (the boundary of a J-dimensional froth is a $(J-1)$-dimensional froth).

As we have seen before, these froths are described by the even valences X_{2l} with $2l < D$. By definition, all the valences X_J must be positive rational numbers and $X_J \geqslant J+1$.

When a valence X_J goes to infinity it means that the J-dimensional cell is covered by an infinite number of $(J-1)$-dimensional simplices. This is a peculiar limit and it can be reached, for example, when the J-dimensional cell becomes infinitely large. The meaning of the limit $X_J \to \infty$ will be different for J odd and J even.

Firstly consider the case of J odd, where, from the equation describing the J-dimensional froth in an elliptic space (equation (11)), the limit $X_J \to \infty$ is reached when the denominator goes to zero, i.e.

$$\left(\sum_{i=0}^{J-2} \frac{(-1)^i}{(J-i)!} \prod_{k=i+1}^{J-1} X_k + 1\right) = 0^+ . \tag{18}$$

This relation between the valences X_k defines a hypersurface in the space of the valences. Note that (18) is identical to (16), which was obtained for a $(J-1)$-dimensional froth covering a Euclidean space.

Now consider a froth on a $(J-1)$-dimensional connected manifold where the Euler's formula gives the following relation between the valences:

$$\left(\sum_{i=0}^{J-2} \frac{(-1)^i}{(J-i)!} \prod_{k=i+1}^{J-1} X_{k+1}\right) C_{J-1} = \chi_{(J-1)} . \tag{19}$$

Here $\chi_{(J-1)}$ is the Euler–Poincaré characteristic of the $(J-1)$-dimensional manifold. Note that the term in the brackets is the same as that in (18) and the sign of this term is the same as $\chi_{(J-1)}$ (because $C_{J-1} > 0$).

Different signs of the Euler–Poincaré characteristic correspond to topologically different spaces. In particular, $\chi_{(J-1)} > 0$ corresponds to an elliptic $(J-1)$-dimensional space and $\chi_{(J-1)} < 0$ correspond to an hyperbolic space, as long as $J-1$ is even (in fact the Gauss–Bonnet theorem, of which Euler's relation is the topological expression, holds only in even-dimensional spaces).

Now, any two regions of the valences' space which have different signs of the bracket term (i.e. of the $\chi_{(J-1)}$) correspond to two froths on manifolds of opposite Gaussian curvature.

Thus the surface in the space of valences defined by (18) is the boundary between sets of manifolds (and froths) of opposite curvature. The points on that surface are therefore froths in a Euclidean $(J-1)$-dimensional space.

Now consider the case of J even, where X_J is a free variable and so $X_J \to \infty$ does not correspond to any restriction of the valences. The reason is that in odd dimensions ($(J-1)$ is odd) the sign of the right-hand side of (19) is not related to space curvature (there is no Gauss–Bonnet theorem in odd dimensions).

Up to $J=5$, from (19) it is straightforward to see explicitly the regions of valence space corresponding to different curvature. In particular, for $J=3$ we have that the region with $X_2 < 6$ corresponds to a positive Euler–Poincaré characteristic and thus to a two-dimensional froth on an elliptic manifold. Conversely, $X_2 > 6$ corresponds to the

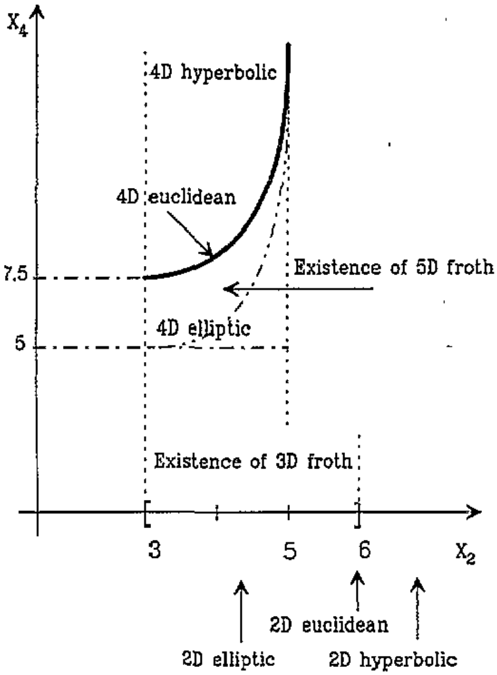

Figure 1. Domains of existence and curvature of froths in dimensions up to $D = 5$.

hyperbolic region and finally $X_2 = 6$ is the Euclidean region (see figure 1). For $J = 5$ we obtain that the curve in the space of the valences $X_4 = 5(6 - X_2)/(5 - X_2)$ corresponds to a four-dimensional froth on a Euclidean manifold, the space below this curve corresponds to the elliptic region and above there is the hyperbolic region (see figure 1).

A J-dimensional froth can only be generated if all the lower-dimensional froths constituting the J-froth are on elliptic manifolds. This yields some restriction on the range of variability of the valences X_{2l}. In particular, all the lower-dimensional valences must be in the range $J + 1 \leqslant X_J < \infty$. When J is odd this condition associated with (11) gives new restriction on the range of variability of the previous even valences.

For $J = D$ even, another restriction on the last valence X_D comes from (17), i.e. from the condition that the D-froth is on an Euclidean manifold.

Up to $D = 5$ the restrictions on the valences are:

$D = 1$: $X_1 = 2$, from (11). This is always valid (any edge is surrounded by two vertices).

$D = 2$: $X_1 = 2$ from (11) and $X_2 = 6$ from (17). This is the well known condition that a two-dimensional Euclidean froth has hexagonal cells on average.

$D = 3$: $X_1 = 2$ and $X_3 = \frac{12}{6-X_2}$.
Furthermore, $4 \leqslant X_3 < \infty$ for the three-dimensional froth to exist, thus $3 \leqslant X_2 < 6$, restricting the free valence X_2.

$D = 4$: $X_1 = 2$ and $X_3 = \frac{12}{6-X_2}$.
The four-dimensional froth is Euclidean if $X_4 = 5\frac{6-X_2}{5-X_2}$ (equation (17)). It only exists if $5 \leqslant X_4 < \infty$, thus $3 \leqslant X_2 < 5$. The upper bound for X_2 is reduced further in $D = 4$ from its value in $D = 3$. For example, the polytopes $\{3, 3, 3, 3\}$ and $\{4, 3, 3, 3\}$ are elliptic regular 'froths', with $X_2 = 3$, $X_4 = 5$ and $X_2 = 4$, $X_4 = 8$, respectively, while $\{5, 3, 3, 3\}$ is a hyperbolic regular 'froth' with $X_2 = 5$ and $X_4 = 120$.

$D = 5$: $X_1 = 2$, $X_3 = \frac{12}{6-X_2}$, $X_5 = 2/\left[1 - \frac{1}{5}X_4\frac{5-X_2}{6-X_2}\right]$.

$6 \leqslant X_5 < \infty$ yields $\frac{10}{3}\frac{6-X_2}{5-X_2} \leqslant X_4 < 5\frac{6-X_2}{5-X_2}$, (the region between full and chain lines in figure 1).

Furthermore, $5 \leqslant X_4 < \infty$ yields $3 \leqslant X_2 < 5$. This necessary condition implies that $\{5, 3, 3, 3, 3\}$ is not a froth, hence that $\{5, 3, 3, 3\}$ is hyperbolic (which is known from [15]).

The region of existence of the five-dimensional froth is contained in (and is smaller than) the region in which the four-dimensional froth is elliptic. Thus the condition of elliptic lower-dimensional froth is a necessary condition for constructing higher-dimensional froths, but it is not sufficient.

5. Classes of froths in all dimensions

The representation in the valences space of a froth of dimensionality J is a succession of free valences $\{X_{2l}\}$ with $2l \leqslant J$. The J-dimensional froth can be, in general, elliptic, Euclidean or hyperbolic. However, for the froth to be constructible in any dimension, it must be elliptic in all dimensions until $J \to \infty$. This implies that the sequence $\{X_{2l}\}$ must satisfy the conditions $J + 1 \leqslant X_J < \infty$ for any J until $J \to \infty$. If, for some finite J, X_J gives a hyperbolic or Euclidean froth, X_{J+2} would be infinite.

There are two interesting sequences that satisfy these two conditions and (11), in any dimension:

$$(\alpha) \qquad X_k = k + 1 \tag{20}$$

and

$$(\beta) \qquad X_k = 2k\,. \tag{21}$$

These expressions are valid both for k even or odd. If k is even, these are the values of the free valences; when k is odd, they are the solutions of (11) for $J = k$, as shown in appendix C.

If one imposes the Euclidean condition (equation (17)) on the even valences for $J = D$, the sequence is interrupted and one gets

$$(\alpha) \qquad X_D = \frac{2}{1 - 2\sum_{i=0}^{D-2}\frac{(-1)^i}{(D+1-i)!}\prod_{k=i+1}^{D-1}(k+1)} = D + 1 + 2\frac{D+1}{D} \tag{22}$$

and for $X_k = 2k$,

$$(\beta) \qquad X_D = \frac{2}{1 - 2\sum_{i=0}^{D-2}\frac{(-1)^i}{(D+1-i)!}\prod_{k=i+1}^{D-1}2k} = 2D + 2 \tag{23}$$

follows from (17) and (21) (see appendix D for the explicit calculation). These values are different from (20) and (21), and approach them only when $D \to \infty$.

5.1. Regular froths

A possible way to construct froths in any dimension is to make them with regular polytopes..

By definition, a froth must be make from polytopes with minimum vertex figures (all vertex figures must be topologically stable). In the Schläfli notation [15] these froths belong to the family $\{X_2, 3^{\delta-2}\}$.

A regular δ-dimensional polytope of this family is a froth on a $(\delta - 1)$-dimensional elliptic manifold (the boundary of the polytope). A Euclidean froth is called honeycomb, after Schläfli.

In any dimension greater than 2, all the possible polytopes and honeycombs of the family $\{X_2, 3^{\delta-2}\}$ are limited in number.

- For $\delta = 2$ one has an infinite number of possible polytopes $\{X_2\}$ and only one possible honeycomb $\{6, 3\}$.
- For $\delta = 3$ one has three possible polytopes: $\{3, 3\}$, $\{4, 3\}$ and $\{5, 3\}$ but no possible honeycomb.
- For $\delta = 4$ one has three polytopes: $\{3, 3, 3\}$, $\{4, 3, 3\}$ and $\{5, 3, 3\}$ and no possible honeycomb.
- For $\delta \geqslant 5$ one has two possible polytopes: $\{3^{\delta-1}\}$, $\{4, 3^{\delta-2}\}$ and no possible honeycomb.

Note that $\{3^{\delta-1}\}$ has $X_k = k + 1$ and $\{4, 3^{\delta-2}\}$ has $X_k = 2k$. These two regular froths are the ordered variants of (i) equation (20) and (ii) equation (21).

Hyperbolic polytopes never occur in natural froths, except in the cubic and sponge phases of amphiphiles [17].

5.2. *Statistical regular froths*

Consider the vertex figure of a D-dimensional regular froth. The number of edges incident on the vertex must be $D + 1$.

Let us assign at every edge a unit vector e_i directed along the edge and pointing out of the vertex.

In a regular froth the vertex figure must be regular and then all the unit vectors e_i must be separated by the same angle. These two conditions correspond to

$$\sum_{i=1}^{D+1} e_i = 0 \tag{24}$$

and

$$e_i e_j = \begin{cases} 1 & \text{if} \quad i = j \\ \cos\alpha_{i,j} = \cos\alpha & \text{if} \quad i \neq j\,. \end{cases} \tag{25}$$

From these equations it follows that

$$\sum_{i,j=1}^{D+1} e_i e_j = D + 1 + \left[(D+1)^2 - (D+1)\right]\cos\alpha = 0 \tag{26}$$

thus,

$$\cos\alpha = -\frac{1}{D}\,.$$

The average number of edges of the two-dimensional facets is

$$X_2 = \frac{2\pi}{\pi - \cos^{-1}\left(-\frac{1}{D}\right)} = \begin{cases} 6 & \text{for} \quad D = 2 \\ 5.104 & \text{for} \quad D = 3 \\ 4.77 & \text{for} \quad D = 4 \\ \vdots & \quad\vdots \\ 4 & \text{for} \quad D \to \infty\,. \end{cases} \tag{27}$$

$X_2 = 6$ in the two-dimensional Euclidean froth is a well known result. In $D = 3$, the value $X_2 = 5.104$ is the same as that obtained by the maximum packing of equal spheres [1, 16].

Finally note that the value $X_2 = 4$, obtained in the limit $D \to \infty$, is the same as that for the β-froth ($X_2 = 2k$) and for the polytope $\{4, 3, 3, \ldots\}$.

6. Conclusions

A D-froth is a graded topological set constituted by D-dimensional polytopes which randomly fill space. The interfaces of these cells are $(D-1)$-elliptic froths, constituted by $(D-1)$-polytopes. Each element of the interface is itself a $(D-2)$-froth etc. The graded topological set has C_k k-dimensional polytopes separated by X_k interfaces each (on average), which are $(k-1)$-froths in elliptic space.

A natural characterization of such a system can be given in terms of the number of k-dimensional simplices and of mean adjacencies numbers $\langle n_{i,j} \rangle$ (with $i < j$). For example, in $D = 2$ these variables are the number of cells, edges, vertices, and the mean number of edges per cell and of vertices per cell and of vertices per edges. In a D-froth these variables are not all independent.

We have shown that a complete topological characterization of a D-froth can be achieved by a set of independent variables: the even 'valences' X_{2k} with $2k = 2, \ldots, 2j < D$. These valences are the average number of the $(2k-1)$-dimensional elements that constitute the interface of a $2k$-dimensional cell. Odd valences are given in terms of the free even valences. We found a canonical form for these relations in spaces of any dimension. The coefficients a_n of these relations have been calculated up to a_5 (corresponding to explicit expressions for the odd valences up to X_{13}). We have found that these coefficients are rapidly converging to $a_\infty = \frac{1}{2}\pi^2$.

A D-froth can be in a Euclidean, hyperbolic or elliptic space, but all its elements are lower-dimensional k-froths in elliptic spaces. It has been shown that the curvature of space influences the range of variability of the free even valences. As the dimension of the froth increases, this range is more restricted.

Two classes of froths (defined by the sequences $X_k = k + 1$ and $X_k = 2k$) have been found which satisfy all the topological conditions in any dimension. These two classes of disordered froths (valences are averages) correspond to the only two possible regular polytopes in spaces of dimension higher than 4. The existence of these two froths in any dimension, the analogy with regular polytopes and the direct calculation of the range of variability for the valences for $D \leqslant 5$, suggest that the sequences $X_k = k + 1$ and $X_k = 2k$ give the limiting values for the range of the free valences in froths of arbitrary dimension.

In summary, these are the main results of this paper:

(i) A topological froth fills space with cells. A D-dimensional froth is a graded set of elliptic topological froths in lower dimensions.
(ii) The incidence numbers are binary coefficients fixed by randomness. The coordination numbers are random variables.
(iii) A topological froth in D-dimensions can be reduced to a problem of linear algebra. There are $[D/2]$ independent variables, $X_{2k} = \langle n_{2k-1,2k} \rangle$, the average number of neighbours of the even-dimensional cells in the froth ($[x]$ is the integer part of x).
(iv) In Schläfli notation, topological froths are $\{X_2, 3, 3, \ldots, 3\}$ for any space curvature, where $3 \leqslant X_2 \leqslant L(D)$. The upper limit is realized $L(3) = 6$, $L(5) = 5$ and it is conjectured that $L(D = \infty) = 4$.

(v) All average coordination numbers have been given in terms of the independent variables X_{2k} by explicit or recursive relations.

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Appendix A. Homological dimension of the froth and number of degrees of freedom

In order to prove the completeness of the set of variables $\{X_2,\ X_4, \ldots,\ X_{2l}, \ldots\}$ it is better to start with relations (4)–(6) and define a new set of variables.

From equation (6), one obtains

$$\langle n_{i,j}\rangle = \binom{D+1-i}{j-i}\frac{C_i}{C_j} = \binom{D+1-i}{j-i}\frac{C_i}{C_\delta}\frac{C_\delta}{C_j} = \binom{D+1-i}{j-i}\frac{\binom{D+1-j}{\delta-j}}{\binom{D+1-i}{\delta-i}}\frac{\langle n_{i,\delta}\rangle}{\langle n_{j,\delta}\rangle}$$
$$\langle n_{i,j}\rangle = \binom{\delta-i}{\delta-j}\frac{\langle n_{i,\delta}\rangle}{\langle n_{j,\delta}\rangle} \qquad \text{(A1)}$$

with $i \leqslant j \leqslant \delta \leqslant D$. Thus, the set of variables $\langle n_{i,\delta}\rangle$ with $i = 1, 2, \ldots, \delta$ is overcomplete.

The relationships between these variables is a system of linear equations, and the number of degrees of freedom can be calculated easily.

Substituting equation (A1) in (5) yields a system of δ linear homogeneous equations in $\delta + 1$ variables:

$$\sum_{i=0}^{j-1}(-1)^i\binom{\delta-i}{j-i}\langle n_{i,j}\rangle - \chi_{j-1}^{(\mathrm{Elliptic})}\langle n_{j,\delta}\rangle = 0 \qquad \text{(A2)}$$

with $j = 1, \ldots, \delta$.

For δ even, the $\delta \times (\delta + 1)$ matrix of coefficents of (A2) has the form

$$\left|\begin{matrix} \binom{\delta}{1} & -2 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 \\ \binom{\delta}{2} & -\binom{\delta-1}{1} & 0 & 0 & 0 & \ldots & 0 & 0 & 0 \\ \binom{\delta}{3} & -\binom{\delta-1}{2} & \binom{\delta-2}{1} & -2 & 0 & \ldots & 0 & 0 & 0 \\ \binom{\delta}{4} & -\binom{\delta-1}{3} & \binom{\delta-2}{2} & -\binom{\delta-3}{1} & 0 & \ldots & 0 & 0 & 0 \\ \vdots & & & & & & & & \vdots \\ \binom{\delta}{\delta-1} & -\binom{\delta-1}{\delta-2} & \binom{\delta-3}{\delta-4} & -\binom{\delta-4}{\delta-5} & \ldots & \ldots & \binom{2}{1} & -2 & 0 \\ \binom{\delta}{\delta} & -\binom{\delta-1}{\delta-1} & \binom{\delta-3}{\delta-3} & -\binom{\delta-4}{\delta-4} & \ldots & \ldots & 1 & -1 & 0 \end{matrix}\right| . \qquad \text{(A3)}$$

The columns go from $j = 0$ to $j = \delta$ and the rows from $i = 1$ to $i = \delta$. Multiplying the generic even row i by $-2/(\delta + 1 - i)$ and adding it to the previous $i - 1$ odd rows reduces this matrix to a triangular form

$$\left|\begin{matrix} 0 & 0 & 0 & 0 & \ldots & & 0 & 0 \\ \binom{\delta}{2} & -\binom{\delta-1}{1} & 0 & 0 & \ldots & & 0 & 0 \\ \vdots & & & & & & \vdots & \\ \vdots & & & & & & \vdots & \\ (\delta-2) & -(\delta-3) & (\delta-4) & -(\delta-5) & \ldots & 0 & 0 & 0 \\ 1 & -1 & 1 & -1 & \ldots & 1 & -1 & 0 \end{matrix}\right| . \qquad \text{(A4)}$$

Moreover, this $\delta \times (\delta + 1)$ triangular matrix has $\delta/2$ zeros on the diagonal. Its rank is $r = \delta - \delta/2 = \delta/2$.

For δ odd, the matrix of coefficents of the system (A2) has exactly the same form as (A3) except for the last row. As before, one has the following triangular matrix:

$$\left|\begin{array}{cccccccc} 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ \binom{\delta}{2} & -\binom{\delta-1}{1} & 0 & 0 & \dots & 0 & 0 & 0 \\ \vdots & & & & & & \vdots & \\ \vdots & & & & & & \vdots & \\ (\delta-1) & -(\delta-2) & (\delta-3) & -(\delta-4) & \dots & -2 & 0 & 0 \\ 1 & -1 & 1 & -1 & \dots & -1 & 1 & -2 \end{array}\right|. \tag{A5}$$

In this case we have a $\delta \times (\delta+1)$ triangular matrix with $(\delta - 1)/2$ zeros on the diagonal. The rank is: $r = (\delta + 1) - (\delta - 1)/2 = (\delta + 1)/2$.

Thus, in any subspace of dimension $\delta < D$, one has $\delta + 1$ variables related by $\delta/2$ or $(\delta + 1)/2$ independent equations for δ even or odd, respectively.

The number of degrees of freedom is therefore

$$\begin{cases} f = \delta + 1 - \frac{\delta}{2} - 1 = \frac{\delta}{2} & \text{for } \delta \text{ even} \\ f = \delta + 1 - \frac{\delta+1}{2} - 1 = \frac{\delta-1}{2} & \text{for } \delta \text{ odd} \end{cases} \tag{A6}$$

where the term '-1' comes from the constraint $\langle n_{\delta,\delta} \rangle = 1$. As for the overcomplete set of variables introduced before, the even valences X_{2l} with $2 \leqslant 2l \leqslant \delta \leqslant D$ are independent because the number of these variables is equal to the number of the degrees of freedom.

When $\delta = D$, one has also the Euler formula for the froth in a Euclidean D-dimensional space (15), which becomes linear and homogeneous if $C_D \to \infty$. This equation is a new relation that must be added to the system of linear equations (A2). For D even, from (15) and (A4), one can write this system of $D+1$ variables and $D+1$ equations as the following $(D + 1) \times (D + 1)$ matrix of coefficents

$$\left|\begin{array}{ccccccc} 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ \binom{D}{2} & -\binom{D-1}{1} & 0 & \dots & 0 & 0 & 0 \\ \vdots & & & & & \vdots & \\ \vdots & & & & & \vdots & \\ (D-2) & -(D-3) & (D-4) & \dots & 0 & 0 & 0 \\ 1 & -1 & 1 & \dots & 1 & -1 & 0 \\ \frac{1}{D+1} & -\frac{1}{D} & \frac{1}{D-1} & \dots & \frac{1}{3} & -\frac{1}{2} & 1 \end{array}\right|. \tag{A7}$$

The last row has the coefficents of Euler's relation. The rank of this matrix is $r = D + 1 - \frac{D}{2} = \frac{D}{2} + 1$. Therefore, for D even, the Euler formula is an additional relation that increases the rank of the matrix by 1.

For D odd, one obtains from (A5) and (15)

$$\left|\begin{array}{ccccccc} 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ \binom{D}{2} & -\binom{D-1}{1} & 0 & \dots & 0 & 0 & 0 \\ \vdots & & & & & \vdots & \\ \vdots & & & & & \vdots & \\ (D-1) & -(D-2) & (D-3) & \dots & -2 & 0 & 0 \\ 1 & -1 & 1 & \dots & -1 & 1 & -2 \\ \frac{1}{D+1} & -\frac{1}{D} & \frac{1}{D-1} & \dots & \frac{1}{3} & -\frac{1}{2} & 1 \end{array}\right|. \tag{A8}$$

In this case, the rank of the matrix is $r = D+1-\frac{D+1}{2} = \frac{D+1}{2}$ and it has not been increased by adding the Euler formula.

Thus, for D even the Euler formula is a new relation between the valences (equation (17)) that decreases the degrees of freedom by one, i.e. $\frac{D}{2}-1$. For D odd the number of degrees of freedom is not modified by the Euler formula, which does not represent a new condition on the variables, so that (17) is equivalent to (11) for $J = D$ odd.

Appendix B. Expression for the dependent valences X_{2s+1} in terms of the independent valences X_{2s}

We want to show, by induction, that X_{2l+1} has the canonical form:

$$X_{2l+1} = \frac{2}{1-\frac{X_{2l}}{a_0}H^1_{2(l-1)}} \tag{B1}$$

where the H^s_{2m} are also expressed by induction as

$$H^s_{2m} = \frac{1-\frac{X_{2m}}{a_s}H^{s+1}_{2(m-1)}}{1-\frac{X_{2m}}{a_0}H^1_{2(m-1)}}. \tag{B2}$$

The induction on H^s_{2m} terminates when the subscript $2m = 0$ ($H^s_0 = 1$, $H^s_{-2} = 0$ for all s). Thus, expression (B2) involves only a finite number of induction steps. The coefficients a_s are numbers, independent of the order $(2l+1)$ of the original valence. For example (see equation (12))

$$\begin{aligned} &a_0 = 6 \qquad a_1 = 5 \qquad a_2 = \frac{84}{17} = 4.94118 \\ &a_3 = \frac{153}{31} = 4.935\,484 \qquad a_4 = \frac{3410}{691} = 4.934877 \\ &a_5 = \frac{26949}{5461} = 4.934\,8105 \qquad a_\infty = \tfrac{1}{2}\pi^2 = 4.934\,8022. \end{aligned} \tag{B3}$$

A general expression (B16) for the a_s will be given below, together with their asymptotic value. Clearly, for $s \geqslant 4$, all the a_s have values very close to their lower bound $a_\infty = \frac{1}{2}\pi^2$.

The expressions for X_1, X_3, X_5 and X_7, given in the text (see equation (12)) have the canonical form (B1) and (B2) with the coefficients $a_0 = 6$, $a_1 = 5$, $a_2 = \frac{84}{17}$ given in (B3). Moreover,

$$H^s_0 = 1 \qquad H^s_2 = \frac{1-\frac{X_2}{a_s}}{1-\frac{X_2}{a_0}} \tag{B4}$$

depend only on the coefficients a_s. After this point, the induction mechanism is set. Assuming that all X_{2j+1} have the canonical form (B1) for all $0 \leqslant j \leqslant l-1$ (that is, assuming also the validity of (B2) for the H^s_{2m}, $0 \leqslant m \leqslant l-2$), we now show that (B1) and (B2) are valid for $j = l$ and $m = l-1$, respectively. In the process of induction, we will obtain a general expression for the coefficients a_s and show that they are independent of the earlier coefficients.

From (11), we have

$$X_{2l+1} = \frac{2}{\left\{1-\frac{X_{2l}}{2}\left[1-\frac{X_{2l-1}}{3}\left(1-\frac{X_{2l-2}}{4}\left(1-\cdots\left(1-\frac{X_1}{2l+1}\right)\right)\right)\right]\right\}} \tag{B5}$$

a function of both dependent (X_{2j+1}) and independent (X_{2j}) valences. Let, by induction, all dependent variable X_{2j+1} take the canonical form (B1) for $0 \leqslant j \leqslant l-1$,

$$X_{2j+1} = \frac{2}{1 - \frac{X_{2j}}{a_0} H^1_{2(j-1)}} \qquad 0 \leqslant j \leqslant l-1 . \tag{B6}$$

Then, using (B2), we have

$$\begin{aligned} X_{2l+1} &= \frac{2}{1 - \frac{X_{2l}}{2}\left[1 - \frac{2}{3}\frac{1-\frac{X_{2l-2}}{4}\left(1-\frac{X_{2l-3}}{5}\left(\ldots\left(1-\frac{X_1}{2l+1}\right)\right)\right)}{1-\frac{X_{2l-2}}{a_0}H^1_{2(l-2)}}\right]} \\ &= \frac{2}{1 - \frac{X_{2l}}{a_0}\frac{N_0}{1-\frac{X_{2l-2}}{a_0}H^1_{2(l-2)}}} \end{aligned} \tag{B7}$$

with

$$N_0 = \frac{a_0}{2}\left(1 - \frac{2}{3}\right) - X_{2l-2}\left\{\frac{H^1_{2(l-2)}}{2} - \frac{2a_0}{4!}\left[1 - \frac{X_{2l-3}}{5}\left(\cdots\left(1 - \frac{X_1}{2l+1}\right)\right)\right]\right\} . \tag{B8}$$

X_{2l+1} has the canonical form if

$$N_0 = 1 - \frac{X_{2l-2}}{a_1} H^2_{2(l-2)} . \tag{B9}$$

Thus $a_0 = 6$, and because X_{2l-3} has by hypothesis the same denominator as $H^1_{2(l-2)}$,

$$N_0 = \frac{a_0}{2}\left(1 - \frac{2}{3}\right) - \frac{X_{2l-2}}{a_1}\frac{N_1}{1 - \frac{X_{2l-4}}{a_0}H^1_{2(l-3)}} \tag{B10}$$

with

$$\begin{aligned} N_1 = {} & \frac{a_1}{2}\left[1 - \frac{2a_0}{3\times 4}\left(1 - \frac{2}{5}\right)\right] \\ & - X_{2l-4}\left\{\frac{H^2_{2(l-3)}}{2} - \frac{2a_1}{4!}H^1_{2(l-3)} + \frac{2^2 a_1 a_0}{6!}\left[1 - \frac{X_{2l-5}}{7}\left(\cdots\left(1 - \frac{X_1}{2l+1}\right)\right)\right]\right\} \\ = {} & 1 - \frac{X_{2l-4}}{a_2} H^3_{2(l-3)} \end{aligned} \tag{B11}$$

to be canonical. Thus, $a_1 = 5$ and $H^2_{2(l-3)}$, $H^1_{2(l-3)}$ and X_{2l-5} have the same denominator: $1 - (X_{2l-6}/a_0)H^1_{2(l-4)}$. By induction hypothesis, we have

$$N_1 = \frac{a_1}{2}\left[1 - \frac{2^2 a_0}{4!}\left(1 - \frac{2}{5}\right)\right] - \frac{X_{2l-4}}{a_2}\frac{N_2}{1 - \frac{X_{2l-6}}{a_0}H^1_{2(l-4)}} \tag{B12}$$

with

$$\begin{aligned} N_2 = {} & \frac{a_2}{2}\left\{1 - \frac{2a_1}{3\times 4}\left[1 - \frac{2a_0}{5\times 6}\left(1 - \frac{2}{7}\right)\right]\right\} - X_{2l-6}\left\{\frac{H^3_{2(l-4)}}{2} - \frac{2a_2}{4!}H^2_{2(l-4)} \right. \\ & \left. + \frac{2^2 a_2 a_1}{6!}H^1_{2(l-4)} - \frac{2^3 a_2 a_1 a_0}{(8!}\left[1 - \frac{X_{2l-7}}{9}\left(\cdots\left(1 - \frac{X_1}{2l+1}\right)\right)\right]\right\} \\ = {} & 1 - \frac{X_{2l-6}}{a_3} H^4_{2(l-4)} . \end{aligned} \tag{B13}$$

This yields immediately $a_2 = \frac{84}{17}$. The process continues till N_{l-1}, i.e. until the subscript of $H_{2[l-(l-1)-2]}$ is equal to -2.

Clearly, the general term is

$$N_k = \frac{a_k}{2}\left\{1 - \frac{2a_{k-1}}{3\times 4}\left[1 - \frac{2a_{k-2}}{5\times 6}\left(1 - \frac{2a_{k-3}}{7\times 8}\left(1 - \cdots\right.\right.\right.\right.$$
$$\left.\left.\left.\left.\cdots\left(1 - \frac{2a_0}{(2k+1)\times(2k+2)}\left(1 - \frac{2}{(2k+3)}\right)\right)\right)\right)\right]\right\}$$
$$-X_{2l-2k-2}\left\{\frac{H^{k+1}_{2(l-k-2)}}{2} - \frac{2a_k}{4!}H^k_{2(l-k-2)} + \cdots\right.$$
$$+(-1)^k\frac{2^k a_k a_{k-1}\cdots a_1}{(2k+2)!}H^1_{2(l-k-2)} - (-1)^k\frac{2^{k+1}a_k a_{k-1}\cdots a_1 a_0}{(2k+4)!}$$
$$\left.\times\left(1 - \frac{X_{2l-2k-3}}{2k+5}\left(\cdots\left(1 - \frac{X_1}{2l+1}\right)\right)\right)\right\}$$
$$= 1 - \frac{X_{(2l-2k-2)}}{a_{k+1}}\frac{N_{k+1}}{1 - \frac{X_{(2l-2k-4)}}{a_0}H^1_{2(l-k-3)}} \tag{B14}$$

in the canonical form. This implies that N_{k+1} has the same form

$$N_{k+1} = \frac{a_{k+1}}{2}\left\{1 - \frac{2a_k}{3\times 4}\left[1 - \frac{2a_{k-1}}{5\times 6}\left(1 - \frac{2a_{k-2}}{7\times 8}\left(1 - \cdots\right.\right.\right.\right.$$
$$\left.\left.\left.\left.\cdots\left(1 - \frac{2a_0}{(2k+3)\times(2k+4)}\left(1 - \frac{2}{(2k+5)}\right)\right)\right)\right)\right]\right\}$$
$$-X_{[2l-2(k+1)-2]}\left\{\frac{a_{k+1}}{2a_{k+1}}H^{k+2}_{2(l-k-3)} - \frac{2a_{k+1}a_k}{4!a_k}H^{k+1}_{2(l-k-3)} + \cdots\right.$$
$$+(-1)^{k+1}\frac{2^{k+1}a_{k+1}a_k\cdots a_1 a_0}{(2k+2)!a_0}H^1_{2(l-k-3)} - (-1)^{k+1}\frac{2^{k+1}a_{k+1}a_k\cdots a_1 a_0}{(2k+4)!}$$
$$\left.\times\left(1 - \frac{X_{2l-2k-5}}{2k+7}\left(\cdots\left(1 - \frac{X_1}{2l+1}\right)\right)\right)\right\} \tag{B15}$$

which demonstrates (B1) by induction.

By the same method, we obtain the series of equations for the coefficients a_s,

$$2 = \frac{2a_k}{2}\left\{1 - \frac{2a_{k-1}}{3\times 4}\left[1 - \frac{2a_{k-2}}{5\times 6}\left(1 - \frac{2a_{k-3}}{7\times 8}\left(1 - \cdots\right.\right.\right.\right.$$
$$\left.\left.\left.\left.\cdots\left(1 - \frac{2a_1}{(2k-1)\times 2k}\left(1 - \frac{2a_0}{(2k+1)\times(2k+2)}\left(1 - \frac{2}{(2k+3)}\right)\right)\right)\right)\right)\right]\right\}. \tag{B16}$$

This yields the sequence given in (B3).

When k is large, the correction to 1 become negligible after some stage, a_k becomes independent of k and satisfies the equation

$$1 + \left\{1 - \frac{2a}{2!} + \frac{(2a)^2}{4!} - \frac{(2a)^3}{6!} + \cdots\right\} = 0 \tag{B17}$$

that is

$$\cos\sqrt{2a} = -1 \tag{B18}$$

so that

$$a_k \simeq \tfrac{1}{2}\pi^2 \qquad \text{for} \quad k \geqslant 4. \tag{B19}$$

Note that (B17), truncated at an even power in the series, has no solutions, whereas if it is truncated at an odd order, its solution is $\leqslant \pi^2/2$. However, the exact equation (B16) always has a solution $a_k \geqslant \pi^2/2$.

Appendix C. Sequences of froths in arbitrary dimensions

Imposing (20) in (11), one obtains

$$\text{(i)} \qquad X_J = \frac{2}{1+\sum_{i=0}^{J-2} \frac{(-1)^i}{(J-i)!} \prod_{k=i+1}^{J-1}(k+1)} = J+1\,. \tag{C1}$$

Indeed,

$$\sum_{i=0}^{J-2} \frac{(-1)^i}{(J-i)!} \prod_{k=i+1}^{J-1}(k+1) = \sum_{i=0}^{J-2} \frac{(-1)^i J!}{(J-i)!(i+1)!} = \frac{1}{(J+1)} \sum_{i=0}^{J-2} (-1)^i \begin{pmatrix} J+1 \\ i+1 \end{pmatrix}$$

$$= -\frac{1}{(J+1)} \sum_{n=0}^{J+1} (-1)^n \begin{pmatrix} J+1 \\ n \end{pmatrix} + \frac{1}{J+1}\left(1+(-1)^J(J+1)-(-1)^J\right)$$

$$= -\frac{1}{J+1}(1-1)^{J+1} + \frac{2-(J+1)}{J+1} \Rightarrow X_J = J+1\,. \qquad \square$$

Imposing (21) in (11), one obtains

$$\text{(ii)} \qquad X_J = \frac{2}{1+\sum_{i=0}^{J-2} \frac{(-1)^i}{(J-i)!} \prod_{k=i+1}^{J-1} 2k} = 2J\,. \tag{C2}$$

Indeed,

$$\sum_{i=0}^{J-2} \frac{(-1)^i}{(J-i)!} \prod_{k=i+1}^{J-1} 2k = \frac{1}{2} \sum_{i=0}^{J-2} \frac{(-1)^i 2^{J-i}(J-1)!}{i!(J-i)!} = \frac{1}{2J} \sum_{i=0}^{J-2} (-1)^i 2^{J-i} \begin{pmatrix} J \\ i \end{pmatrix}$$

$$= \frac{1}{2J} \sum_{n=0}^{J} (-1)^n 2^{J-n} \begin{pmatrix} J \\ n \end{pmatrix} - \frac{1}{2J}\left(-(-1)^J 2J + (-1)^J\right)$$

$$= \frac{1}{2J}(-1+2)^J + \frac{1-2J}{2J} \Rightarrow X_J = 2J\,. \qquad \square$$

Appendix D. Euclidean termination of the froth sequences

The Euclidean condition (equation (17)) on the even valence X_D yields

$$\text{(i)} \qquad X_D = \frac{2}{1-2\sum_{i=0}^{D-2} \frac{(-1)^i}{(D+1-i)!} \prod_{k=i+1}^{D-1}(k+1)} = \frac{(D+1)(D+2)}{D}\,. \tag{D1}$$

Indeed, if we put formaly $J = D+1$ in the identity (C1), we obtain

$$\sum_{i=0}^{D-1} \frac{(-1)^i}{(D+1-i)!} \prod_{k=i+1}^{D} (k+1) = -1 + \frac{2}{D+2} = -\frac{D}{D+2}$$

$$\left(\sum_{i=0}^{D-2}\frac{(-1)^i}{(D+1-i)!}\prod_{k=i+1}^{D-1}(k+1)+\frac{(-1)^{D-1}}{2!}\right)(D+1)=-\frac{D}{D+2}$$

$$\Rightarrow \sum_{i=0}^{D-2}\frac{(-1)^i}{(D+1-i)!}\prod_{k=i+1}^{D-1}(k+1)=\frac{1}{2}-\frac{D}{(D+1)(D+2)}$$

$$\Rightarrow X_D=\frac{(D+1)(D+2)}{D}. \qquad \square$$

For $X_k = 2k$,

(ii) $$X_D=\frac{2}{1-2\sum_{i=0}^{D-2}\frac{(-1)^i}{(D+1-i)!}\prod_{k=i+1}^{D-1}2k}=2(D+1) \qquad \text{(D2)}$$

follows from (17) and (21).

As above,

$$\sum_{i=0}^{D-2}\frac{(-1)^i}{(D+1-i)!}\prod_{k=i+1}^{D-1}2k=\frac{1}{4D(D+1)}\sum_{i=0}^{D-2}\frac{(-1)^i 2^{D+i}(D+1)!}{i!(D+1-i)!}$$

$$=\frac{(-1+2)^{D+1}}{4D(D+1)}$$

$$-\frac{1}{4D(D+1)}\left((-1)^{(D+1)}2^D\frac{D(D+1)}{2}+(-1)^D 2(D+1)+(-1)^{D+1}\right)$$

$$=\frac{D}{2(D+1)}\Rightarrow X_D=2(D+1). \qquad \square$$

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